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The Gelfand-Naimark Theorem for C^*Algebras Over Arens Algebras

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ABSTRACT

In this paper we prove a vector version of Gelfand-Naimark theorem for C^*-algebra over Arens algebra.

Keywords: C^*algebra over Arens algebras, Gelfand-Naimark theorem, commutative algebra.

1. Introduction

The Gelfand-Naimark theorem is the one of the most important results in the theory of Banach algebras. The development of the general theory of Banach-Kantorovich C*-algebras over the ring of measurable functions naturally leads to the question about an analog of the Gelfand-Naimark theorem for such C*-modules. The theory of C*-modules comes from the work I.Kaplansky (Kaplasky, 1953). In Kusraev (1996) it was proved a vector a vector valued version of Gelfand-Mazur's theorem for C*-modules over Stone algebra The theory of Banach-Kantorovich modules is being actively developed now (Kusraev, 2003),(Gutman, 1995). In (Ganiev and Chilin, 2003) a C*-algebra over a ring of measurable functions as a measurable bundle of a classic C*-algebras is presented. In Chilin et al. (2008) the Gelfand-Naimark theorem for C*-algebras over a ring of measurable functions is proved.

In this paper we are going to prove the Gelfand-Naimark theorem for C^*-algebras over Arens algebras.

2. Preliminaries

Let (Ω, Σ, μ) be a measurable space with a complete finite measure and let $L^0 = L^0(\Omega)$ be the algebra of all complex measurable functions defined on (Ω, Σ, μ), E be a complex linear space.

The mapping $\|\cdot\| : E \to L^0$ is called an L^0-valued norm on E, if for any $x, y \in E$, $\lambda \in C$ satisfies the following

$$\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0, \|\lambda x\| = |\lambda|\|x\|, \|x + y\| \leq \|x\| + \|y\|.$$

A pair $(E, \|\cdot\|)$ is called a lattice-normed space (LNS) over L^0. An LNS E is said to be d-decomposable, if for any $x \in E$ and for any decomposition $\|x\| = e_1 + e_2$ into a sum of disjunctive elements one can find $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_1\| = e_1, \|x_2\| = e_2$. A net $(x_\alpha)_{\alpha \in A}$ of elements of E is said to be (bo)-converging to $x \in E$, if the net $\|x_\alpha - x\|_{\alpha \in A}$ (o)-converges to zero in L^0. A (bo)-complete d-decomposable LNS over L^0 is called a Banach-Kantorovich space (BKS) over L^0 ((Kusraev, 1985), P. 32; (Kusraev, 2003), P. 79).

Let U be an arbitrary *-algebra over the field $\mathbb{C}$ of complex numbers and let U be the module over L^0; assume that $(\lambda u)^* = \overline{\lambda} u^*, (\lambda u)v = \lambda(uv) = u(\lambda v)$ for all $\lambda \in L^0, u, v \in U$. Consider on U a certain L^0-valued norm $\|\cdot\|$, endowing U with the structure of a Banach-Kantorovich space, in particular, $\|\lambda u\| = |\lambda|\|u\|$ for all $\lambda \in L^0, u \in U$.

Definition 2.1. *(Kusraev, 1985) U is called a C^*-algebra over L^0, if all $u, v \in U$ satisfies $\|u \cdot v\| \leq \|u\|\|v\|, \|u\|^2 = \|u^* \cdot u\|$. If U is a C^*-algebra over L^0 with the unit e and $\|e\| = \mathbf{1}$, where $\mathbf{1}$ is the unit in L^0, then U is called a unital C^*-algebra over L^0.*

Let X be a mapping, which sends every point $\omega \in \Omega$ to some C^*-algebra

$(X(\omega), \|\cdot\|_{X(\omega)})$. We assume that $X(\omega) \neq \{0\}$ for all $\omega \in \Omega$. A function u is said to be a section of X, if it is defined almost everywhere in Ω and takes values $u(\omega) \in X(\omega)$ for $\omega \in dom(u)$, where $dom(u)$ is the domain of u.

Let L be some set of sections.

Definition 2.2. *(Kusraev, 1985) A pair $(\mathcal{X}, L)$ is called a measurable bundle of C^*-algebras, if*

1. $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $c_1, c_2 \in L$, where $\lambda_1 c_1 + \lambda_2 c_2 : \omega \in dom(c_1) \bigcap dom(c_2) \to \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;

2. the function $\|c\| : \omega \in dom(c) \to \|c(\omega)\|_{U(\omega)}$ is measurable for all $c \in L$;

3. for each point $\omega \in \Omega$ the set $\{c(\omega) : c \in L, \omega \in dom(c)\}$ is dense in $U(\omega)$;

4. if $u \in L$, then $u^* \in L$, where $u^* : \omega \in dom(u) \to u(\omega)^*$;

5. if $u, v \in L$, then $u \cdot v \in L$, where $u \cdot v : \omega \in dom(u) \bigcap dom(v) \to u(\omega) \cdot v(\omega)$.

A section s is called stepwise, if $s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega) c_i(\omega)$, where $c_i \in L, A_i \in \Sigma, i = \overline{1, n}$. A section u is called measurable, if one can find a sequence $(s_n)_{n \in \mathbb{N}}$ of stepwise sections such that $\|s_n(\omega) - u(\omega)\|_{U(\omega)} \to 0$ for almost all $\omega \in \Omega$.

The set of all measurable sections is denoted by $M(\Omega, X)$, and $L^0(\Omega, X)$ denotes the factorization of this set with respect to equality almost everywhere on Ω. We denote by $\widehat{u}$ the class from $L^0(\Omega, X)$ containing a section $u \in M(\Omega, X)$, and by $\|\widehat{u}\|$ the element of L^0 containing the function $\|u(\omega)\|_{X(\omega)}$.

Put $\hat{u} \cdot \hat{v} = u(\omega) \hat{\cdot} v(\omega)$ and $\hat{u}^* = u(\hat{\omega})^*$.It is shown in Kusraev (1985) that with respect to the introduced algebraic operations $(L^0(\Omega, X), \|\cdot\|)$ is a C^*-algebra over L^0.

Let $\mathcal{L}^\infty(\Omega)$ be an algebra of bounded measurable functions on (Ω, Σ, μ); let $L^\infty(\Omega)$ be a factorization of $\mathcal{L}^\infty(\Omega)$ with respect to the equality a. e. Put $\mathcal{L}^\infty(\Omega, X) = \{u \in M(\Omega, X) : \|u(\omega)\|_{U(\omega)} \in \mathcal{L}^\infty(\Omega)\}$. Elements of $\mathcal{L}^\infty(\Omega, X)$ are said to be essentially bounded measurable sections. By $L^\infty(\Omega, X)$ we denote the set of equivalence classes of essentially bounded sections.

Consider an arbitrary lifting $p : L^\infty(\Omega) \to \mathcal{L}^\infty(\Omega)$ ((Kusraev, 1985), P. 50;

(Chilin et al., 2008)).

Definition 2.3. *(Kusraev, 1985) A mapping* $l_\chi : L^\infty(\Omega, X) \to \mathcal{L}^\infty(\Omega, X)$ *is called a vector-valued lifting (associated with the lifting p), if all* $\hat{u}, \hat{v} \in L^\infty(\Omega, X)$ *and* $\lambda \in L^\infty(\Omega)$ *satisfy the following correlations:*

1. $l_\chi(\hat{u}) \in \hat{u}, dom l_\chi(\hat{u}) = \Omega$;
2. $\|l_\chi(\hat{u})(\omega)\|_{U(\omega)} = p(\|\hat{u}\|)(\omega)$;
3. $l_\chi(\hat{u} + \hat{v}) = l_\chi(\hat{u}) + l_\chi(\hat{v})$;
4. $l_\chi(\lambda\hat{u}) = p(\alpha) l_\chi(\hat{u})$;
5. $l_\chi(\hat{u}^*) = l_\chi(\hat{u})^*$;
6. $l_\chi(\hat{u}\hat{v}) = l_\chi(\hat{u}) l_\chi(\hat{v})$;
7. *the set* $\{l_\chi(\hat{u})(\omega) : \hat{u} \in L^\infty(\Omega, X)\}$ *is dense in* $U(\omega)$ *for all* $\omega \in \Omega$.

It is well-known ((Kusraev, 1985), theorem 2) that for any C^*-algebra U over L^0 a measurable bundle of C^*-algebras (X, L) exists such that U is isometrically *-isomorphic to $L^0(\Omega, X)$, and on $L^\infty(\Omega, X)$ a lifting exists which is associated with a certain numerical lifting p.

A functional $f : U \to L^0$ is called L^0-linear, if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $\alpha, \beta \in L^0, x, y \in U$. An L^0-linear functional $f : U \to L^0$ is called L^0-bounded, if one can find $c \in L^0$ such that $\|f(x)\| \leq c\|x\|$ for all $x \in U$. For an L^0-linear L^0-bounded functional $f : U \to L^0$ we put $\|f\| = sup\{|f(x)| : x \in U, \|x\| \leq \mathbf{1}\}$. An L^0-linear functional $f : U \to L^0$ is called positive ($f \neq 0$), if $f(xx^*) \neq 0$ for all $x \in U$.

The mentioned functional is called a state, if $f \neq 0$ and $\|f\| = \mathbf{1}$.

A state φ is called pure, if the relation $\varphi \neq \psi \neq 0$, where ψ is an L^0-linear functional, implies that $\psi = \lambda\varphi$ for certain $\lambda \in L^0, 0 \leq \lambda \leq \mathbf{1}$.

For $p \in [1; \infty]$ we denote

$$L^p(\Omega, X) = \{\hat{u} \in L^0(\Omega, X) : \|u(\omega)\|_{X(\omega)} \in L^p\}.$$

From Bekbaev and Ganiev (2014) we know that $L_p(\Omega, X)$ is Banach space with respect to the norm:

$$\|\hat{u}\|_{L^p(\Omega,X)} = \left(\int\limits_{\Omega} \|u(\omega)\|^p_{X(\omega)} d\mu\right)^{1/p} = \|\|u(\omega)\|_{X(\omega)}\|_{L^p(\Omega,X)}$$

Let

$$L^{\omega}(\Omega, X) = \bigcap_{p\geq 1} L^p(\Omega, X)$$

i.e.

$$L^{\omega}(\Omega, X) = \{\hat{u} \in L^0(\Omega, X) : \|\hat{u}\|_1 < \infty, \|\hat{u}\|_2 < \infty, ..., \|\hat{u}\|_p < \infty, ...\}.$$

We will consider in $L^{\omega}(\Omega, X)$ locally convex topology τ_X is generated by system of norms $\{\|\cdot\|_{L^p(\Omega,X)}\}_{p\geq 1}$. We know from Bekbaev and Ganiev (2014) that

$$\|\hat{u}\|_{L^1(\Omega,X)} \leq \|\hat{u}\|_{L^2(\Omega,X)} \leq \cdots \leq \|\hat{u}\|_{L^p(\Omega,X)} \leq \cdots,$$

i.e. the topology τ_X generated by countable system of norms $\{\|\cdot\|_{L^n(\Omega,X)}\}_{n=1}^{\infty}$. By Theorem III.2.2 (Kantorovich and Akilov, 1982) it means that topological vector space $(L^{\omega}(\Omega, X), \tau_X)$ is metrizable space with respect to the metric

$$d(\hat{u}, \hat{v}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|\hat{u} - \hat{v}\|_{L^k(\Omega,X)}}{1 + \|\hat{u} - \hat{v}\|_{L^k(\Omega,X)}}.$$

3. The State Space of C*-Algebras Over Arens Algebras

Let a unital C^*-algebra U over L^{ω}. We assume that U has the form $L^{\omega}(\Omega, X)$, where X is a measurable bundle of C^*-algebras with a vector-valued lifting.

If $\varphi \neq 0, a, b \in U, \lambda \in L^{\omega}$, then $\varphi((\lambda a + b)^*(\lambda a + b)) \neq 0$, i.e., $|\lambda|^2\varphi(a^*a) + \overline{\lambda}\varphi(a^*b) + \lambda\varphi(b^*a) + \varphi(b^*b) \neq 0$. Therefore

$$\varphi(a^*b) = \overline{\varphi(b^*a)}, |\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b). \qquad (1)$$

Consequently, for $\varphi \neq 0, a \in U$ we have $\varphi(a^*) = \overline{\varphi(a)}$; in addition, $\varphi = 0$, if $\varphi(e) = 0$.

The following two propositions are versions of the well-known properties of positive functionals of C^*-algebras for the case of C^*-algebras over L^ω.

Proposition 3.1. *Let U^* be the set of all L^ω-linear L^ω-bounded functionals on U. Then*

(a) if $\varphi \neq 0$, then $|\varphi(x)|^2 \leq \varphi(e)\varphi(x^*x) \leq \varphi(e)^2\|x\|^2$, in particular, $\varphi \in U^*$ and $\|\varphi\| = \varphi(e)$;

(b) if $\varphi \in U^*$ and $\|\varphi\| = \varphi(e)$, then $\varphi \neq 0$;

(c) if $\varphi \in U^*$ and $\|\varphi\| = \mathbf{1} = \varphi(e)$, then φ is a state;

(d) if $\varphi, \psi \neq 0$ and $\alpha, \beta \in L^\omega, \alpha, \beta \neq 0$ then $\alpha\varphi + \beta\psi \neq 0$ and $\|\alpha\varphi + \beta\psi\| = \alpha\|\varphi\| + \beta\|\psi\|$, the set E_U of all states on U is a convex set.

Proof. (a) Since $(L^\infty(\Omega, X), \|\cdot\|)$ is a BKS over $L^\infty(\Omega)$, we conclude that $(L^\infty(\Omega, X), \|\cdot\|_\infty)$ is a Banach space with respect to the norm $\|x\|_\infty = \|\|x\|\|_{L^\infty(\Omega)}, x \in L^\infty(\Omega, \mathcal{X})$. It is known that $(L^\infty(\Omega, X), \|\cdot\|_\infty)$ is a C^*-algebra over $\mathbb{C}.((\mathbf{?}))$.

Let $x \in U$. For any $n \in \mathbb{N}$ put $\Omega_n = \{\omega \in \Omega : \|x\|(\omega) < \frac{1}{n}\}$. We define function α_n by following formula:

$$\alpha_n(\omega) = \begin{cases} 0, & \omega \in \Omega_n; \\ \frac{1}{\|x\|(\omega)}, & \omega \in \Omega_n. \end{cases}$$

It is easy to check that $\alpha_n \in L^\omega$ for all n.

Put $z_n = \alpha_n x, n \in \mathbb{N}$. Then $\|z_n^* z_n\| = \|\alpha_n x\|^2 = |\alpha_n|^2\|x\|^2 = \pi_n \leq \mathbf{1}$, where $\pi_n = \chi_{\Omega\setminus\Omega_n}$.

Hence $\|z_n^* z_n\|_\infty \leq 1$, so $e - z_n^* z_n$ is a positive element in $L^\infty(\Omega, X)$, i.e. one can find $u \in L^\infty(\Omega, X)$ such that $e - z_n^* z_n = u_n^* u_n$. From here $\varphi(e) - \varphi(z_n^* z_n) = \varphi(u_n^* u_n) \geq 0$ and $\varphi(z_n^* z_n) \leq \varphi(e)$.

Set $x_n = \|x\| z_n$ and therefore

$$\varphi(x_n^* x_n) = \|x\|^2 \varphi(z_n^* z_n) \leq \varphi(e)\|x\|^2$$

so

$$\varphi(x_n^* x_n) \leq \varphi(e)\|x\|^2.$$

Now using inequality (1) we get

$$|\varphi(x_n)|^2 = |\varphi(ex_n)|^2 \leq |\varphi(e^*e)\varphi(x_n^*x_n) \leq \varphi(e)^2\|x\|^2$$

which means

$$|\varphi(x_n)| \leq \varphi(e)\|x\| \tag{2}$$

As $x_n = \|x\|z_n = \|x\|\alpha_n x$ we get

$$x_n = \|x\|\alpha_n x = \pi_n x. \tag{3}$$

Combining (2) and (3) we get $|\varphi(\pi_n x)| \leq \varphi(e)\|x\|$ and $\pi_n|\varphi(x)| \leq \varphi(e)\|x\|$

Since $\pi_n \uparrow \mathbf{1}$ we get

$$|\varphi(x)| \leq \varphi(e)\|x\|. \tag{4}$$

From inequality (4) we get $\varphi \in U^*$ and $\|\varphi\| \leq \varphi(e)$. Since $\|e\| = \mathbf{1}$ we get $\|\varphi\| = \varphi(e)$.

(b) Let $\varphi \in U^*$ and $\|\varphi\| = \varphi(e)$. With no loss of generality, assume that $\|\varphi\| = \varphi(e) \in L^\infty(\Omega)$ (otherwise we consider $\frac{\varphi}{1+\|\varphi\|}$).For each $\omega \in \Omega$ we define a $\mathbb{C}$-linear functional φ_ω on $L^\infty(\Omega, X)$ by the following rule:

$$\varphi_\omega(x) = p(\varphi(x))(\omega),\, x \in L^\infty(\Omega, \mathcal{X}), \tag{5}$$

where p is a numerical lifting on $L^\infty(\Omega)$. Let $\|x\|_\infty \leq 1$. Then $\|x\| \leq \mathbf{1}$ and

$$|\varphi_\omega(x)| = p(|\varphi(x)|)(\omega) \leq p(\varphi(e))(\omega) = \varphi_\omega(e).$$

Consequently, φ_ω is a bounded functional on $L^\infty(\Omega, X)$, and $\|\varphi_\omega\| \leq \varphi_\omega(e)$. As $\|e\|_\infty = \mathbf{1}$, we get $\|\varphi_\omega\| = \varphi_\omega(e)$ for all $\omega \in \Omega$. This means that $\varphi_\omega \geq 0$ for all $\omega \in \Omega$, and therefore $\varphi \geq 0$.

Items (c) and (d) immediately follow from (b). □

Proposition 3.2. *Let U be a C^*-algebra over L^ω, and E_U be the set of all states on $U, \varphi \in E_U$. Consider the following conditions:*

(a) φ is an extreme point of E_U ;

(b) φ is a pure state;

(c) φ is a homomorphism, i. e., $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in U$.

Then $(a) \Leftrightarrow (b)$, and if U is commutative, then conditions (a), (b), (c) are equivalent.

Proof. $(a) \Rightarrow (b)$ Let $\varphi \in E_U$ be an extreme point. Assume that φ is not pure. Then one can find $\psi \geq 0$ such that $\varphi \geq \psi$ and $\psi \neq \lambda\varphi$ for all $\lambda \in L^\omega, 0 \leq \lambda \leq \mathbf{1}$. Since $\mathbf{1} = \varphi(e) \geq \psi(e) \geq 0$, we have $\psi \neq \psi(e)\varphi$. Assume that $\psi(e) = \chi_A$ for certain $A \in \Sigma$. Then $\chi_A\psi = \chi, \chi_A(\varphi - \psi) \geq 0$, and $0 \leq \chi_A(\varphi(e) - \psi(e)) = \chi_A - \chi_A = 0$. Hence $\chi_A\psi = \chi_A\varphi$ and $\psi = \chi_A\varphi = \psi(e)\varphi$, what contradicts the inequality $\psi \neq \psi(e)\varphi$. Therefore there exists $0 < t < 1$ such that the set $B = \{\omega \in \Omega : t < \psi(e)(\omega) < 1\}$ has a positive measure, i.e., $\pi = \chi_B \neq 0$. Let us define $\alpha \in L^\omega$, putting

$$\alpha(\omega) = \begin{cases} 0, & w \in B \\ \frac{1}{\psi(e)(w)}, & w \in B \end{cases}$$

Then $t\alpha \leq \pi, \varphi_1 = \alpha\psi + \pi^\perp\varphi$ is a state, where $\pi^\perp = 1 - \pi$, and $t\varphi_1 = t\alpha\psi + t\pi^\perp\varphi \leq \pi\psi + \pi^\perp\varphi \leq \varphi$.

Put $\varphi_2 = \frac{\varphi - t\varphi_1}{1-t}$. Clearly, $\varphi \geq \varphi_2 \geq 0$ and $\|\varphi_2\| = \frac{\|\varphi\| - t\|\varphi_1\|}{1-t} = \mathbf{1}$, i.e., φ_2 is also a state, in addition, $\varphi_1 = t\varphi_1 + (1-t)\varphi_2$, what contradicts condition (a).

$(b) \Rightarrow (a)$ Let φ be a pure state and $\varphi = t\varphi_1 + (1-t)\varphi_2$, where $0 < t < 1, \varphi_1, \varphi_2 \in E_U$. Then $\varphi \geq t\varphi_1$, and therefore $t\varphi_1 = \lambda\varphi$ with certain $\lambda \in L^\omega, 0 \leq \lambda \leq \mathbf{1}$, in particular, $t\mathbf{1} = t\varphi_1(e) = \lambda\varphi(e) = \lambda$. Consequently, $\varphi = \varphi_1$, and therefore φ is a limiting point.

Now let U be a commutative C^*-algebra over L^ω.

$(b) \Rightarrow (c)$. Let $\varphi \in E_U$ be a pure state. Let us first prove the following

$$\varphi(xx^*y) = \varphi(xx^*)\varphi(y), x, y \in U \tag{6}$$

Let $x, u \in L^{\infty}(\Omega, \mathcal{X}), \|xx^*\|_{\infty} < 1, e - xx^* = uu^*$. We will put

$$\psi(y) = \varphi(xx^*y), y \in U.$$

Using positivity φ, we get $\psi(yy^*) = \varphi(xx^*yy^*) = \varphi((xy)(xy)^*) \geq 0$ and $\varphi(yy^*) - \psi(yy^*) = \varphi(yy^*) - \varphi(xx^*yy^*) = \varphi((e - xx^*)yy^*) = \varphi(uu^*yy^*) = \varphi((uy)(uy)^*) \geq 0$. Hence $\varphi \geq \psi \geq 0$. As φ is a pure state, we obtain $\psi = \psi(e)\varphi$. Consequently, $\varphi(xx^*y) = \psi(y) = \psi(e)\varphi(y) = \varphi(xx^*)\varphi(y)$.

The fact that (6) implies (c) follows from the evident identity $x = \frac{1}{3}\sum_{k=1}^{3} \theta^k z_k z_k^*$,where $\theta = exp(2\pi i/3), z_k = e - \theta^{-k}, k = 1, 2, 3$, for all $x \in U$.

$(c) \Rightarrow (b)$ Let φ be a homomorphism from U to L^{ω} and $\varphi \geq \psi \geq 0$. Then $\varphi_{\omega}, \psi_{\omega}$ are positive $\mathbb{C}$-linear functionals on $L^{\infty}(\Omega, \mathcal{X})$ and $\varphi_{\omega} \geq \psi_{\omega}$ for all $\omega \in \Omega$; in addition, φ_{ω} is a homomorphism from $L^{\infty}(\Omega, \mathcal{X})$ into $\mathbb{C}$, where $\varphi_{\omega}, \psi_{\omega}$ are defined by correlation (5). By Bratteli and Robinson (1982) (P. 67) we get that φ_{ω} is a pure numerical state on $L^{\infty}(\Omega, \mathcal{X})$. Hence $\psi_{\omega} = \psi_{\omega}(e)\varphi_{\omega}$ for all $\omega \in \Omega$, and therefore

$$\psi = \varphi(e)\varphi$$

□

For $\varepsilon > 0$ we put

$$W(\varepsilon) = \{\lambda \in L^{\omega} : \rho(\lambda, 0) < \varepsilon\}.$$

The system $\mathbb{W} = \{W(\varepsilon) : \varepsilon > 0\}$ generates in L^{ω} a natural topology; in addition, $\mathbb{W}$ is a base of zero neighborhoods in this topology.

Consider in U^* a separable vector $*$-weak topology, whose base of zero neighborhoods is represented by sets in the form

$$V\langle \varepsilon, \delta, x_1, ...x_n \rangle = \{f \in U^* : |f(x_i)| \in W(\varepsilon), i = \overline{1, n}\},$$

where $x_1, ..., x_n \in U, \varepsilon > 0$.

By ∇ we define a Boolean algebra of all idempotents in L^{ω} and let $F \subset U^*$. If $(u_{\alpha})_{\alpha \in A} \subset F$, and $(\pi_{\alpha})_{\alpha \in A}$ is a unity partition in ∇ and the series $\sum_{\alpha \in A} \pi_{\alpha} u_{\alpha}$ *-weakly converges, then the sum of this series is called a confusion of $(u_{\alpha})_{\alpha \in A}$ with respect to $(\pi_{\alpha})_{\alpha \in A}$. This sum is denoted by $mix(\pi_{\alpha} u_{\alpha})$. For F the symbol $mixF$ stands for the set of all confusions of arbitrary families of elements of F . The set F is called cyclic, if $mixF = F$. For a directed set

A the symbol $\nabla(A)$ stands for the set of all unity confusions in ∇ indexed by elements of A. Define the ordering relation on $\nabla(A)$ as follows:

$$v_1 \leq v_2 \Leftrightarrow \forall \alpha, \beta \in A, (v_1(\alpha) \wedge v_2(\beta) \neq 0 \rightarrow \alpha \leq \beta)(v_1, v_2 \in \nabla(A)).$$

Let $(u_\alpha)_{\alpha \in A}$ be a net in F . For each $v \in \nabla(A)$ we put $u_v = mix(v(\alpha)u_\alpha)$ and obtain a new net $(u_v)_{v \in \nabla(A)}$. An arbitrary subnet of the net $(u_v)_{v \in \nabla(A)}$ is called a cyclic subnet of the net $(u_\alpha)_{\alpha \in A}$. The set $F \subset U^*$ is said to be *-weakly cyclically compact (Kusraev, 1985), if it is cyclic and each net in F has a cyclic subnet, *-weakly converging to a certain point of F ((Kusraev, 1985), P. 50).

Proposition 3.3. *Let U be a C^*-algebra over L^ω. Then*

*(a) E_U is *-weakly cyclically compact;*

*(b) if the algebra U is commutative, then the set $K(U)$ of all pure states on U is *-weakly cyclically compact.*

Proof. (a) As $E_U \subset U_1^*$ and U_1^* is *-weakly cyclically compact, it suffices to prove that E_U is a cyclic and *-weakly closed subset in U_1^*. As the measure μ is finite, we can consider only countable partitions in ∇.

Let $(\pi_n)_{n \in \mathbb{N}}$ be an arbitrary unity partition in $\nabla, (\varphi_n)_{n \in \mathbb{N}} \subset K(U)$ and $\varphi = \sum_{n=1}^{\infty} \pi_n \varphi_n$. Then $\varphi_n(xx^*) \geq 0$ and $\varphi_n(e) = \mathbf{1}$ for all $x \in U, n \in \mathbb{N}$. Therefore $\varphi(xx^*) = \sum_{n=1}^{\infty} \pi_n \varphi_n(xx^*) \geq 0$ and $\varphi(e) = \sum_{n=1}^{\infty} \pi_n \varphi_n(e) = \sum_{n=1}^{\infty} \pi_n = \mathbf{1}$. Due to **Proposition 3.1** we have $\varphi \in E_U$.

If $\varphi \in U_1^*$ belongs to an *-weak closure of E_U, then one can find a net $\{\varphi_\alpha\}$ in E_U such that $\{\varphi_\alpha(x)\}$ converges in L^ω by norm to $\varphi(x)$ for all $x \in U$. Consequently we obtain $\varphi(xx^*) \geq 0$ and $\varphi(e) = \mathbf{1}$ for any $x \in U$. That is means $\varphi \in E_U$.

(b) It suffices to prove that $K(U)$ is a cyclic *-weakly closed subset in E_U.

Let $(\pi_n)_{n \in \mathbb{N}}$ be an arbitrary unity partition in $\nabla, (\varphi_n)_{n \in \mathbb{N}} \subset K(U)$ and $\varphi = \sum_{n=1}^{\infty} \pi_n \varphi_n$. Since the algebra U is commutative, by **Proposition 3.2** φ_n is a homomorphism for each $n \in \mathbb{N}$. Therefore for all $x, y \in U$ we have

$$\varphi(xy) = \sum_{n=1}^{\infty} \pi_n \varphi_n(xy) = \sum_{n=1}^{\infty} \pi_n \varphi_n(x)\varphi_n(y) = \sum_{n=1}^{\infty} \pi_n \varphi_n(x) \sum_{n=1}^{\infty} \pi_n \varphi_n(y) = \varphi(x)\varphi(y)$$

Which means φ is a homomorphism, and therefore $\varphi \in K(U)$ (see **Proposition 3.2**).

Let now φ belong to an *-weak closure of $K(U)$ and $\{\varphi_\alpha\}$ be a net in $K(U)$ such that $\{\varphi_\alpha(x)\}$ converges in L^ω to $\varphi(x)$ for all $x \in U$. By **Proposition 3.2** we have $\varphi_\alpha(xy) = \varphi_\alpha(x)\varphi_\alpha(y)$ for all α and so $\varphi(xy) = \varphi(x)\varphi(y)$, i.e., φ is a homomorphism. Using **Proposition 3.2** once again, we obtain $\varphi \in K(U)$. $\square$

Proposition 3.4. *Let U be a commutative C^*-algebra over L^ω and $a \in U$. Then on U there exists $\varphi \in K(U)$ such that $\varphi(a^*a) = \|a\|^2$.*

Proof. Let us assume that $\|a\| \in L^\infty(\Omega)$ (otherwise consider $\frac{a}{\mathbf{1}+\|a\|}$).Let $B = \{\alpha e + \beta a^*a : \alpha, \beta \in L^\infty(\Omega)\}$. On B we define an $L^\infty(\Omega)$-valued functional f by the rule

$$f(\alpha e + \beta a^*a) = \alpha + \beta\|a\|^2(\alpha, \beta \in L^\infty(\Omega)).$$

Let us check is f defined correctly.

Case 1. Elements $\{e, a^*a\}$ are ∇-linearly independent. Therefore for any $\pi \in \nabla$ and $\lambda_1, \lambda_2 \in L^\omega$ the formula $\pi(\lambda_1 e + \lambda_2 a^*a) = 0$ yields $\pi\lambda_1 = \pi\lambda_2 = 0$ ((Kusraev, 1985), P. 197). In this case the element $\alpha e + \beta a^*a$ is uniquely defined in terms of α, β. Consequently $f(\alpha e + \beta a^*a) = \alpha + \beta\|a\|^2$ is uniquely defined in terms of α, β.

Case 2. Elements $\{e, a^*a\}$ are ∇-linearly dependent. With no loss of generality, assume that $a^*a = \lambda e, \lambda \in L^\infty(\Omega), \lambda \geq 0$. Then $f(\alpha e + \beta a^*a) = \alpha + \beta\|a\|^2 = \alpha + \beta\lambda$, and in this case f is defined correctly.

Fix $\omega \in \Omega$ and $\alpha, \beta \in L^\infty(\Omega)$. Put $\alpha_\omega = p(\alpha)(\omega), \beta_\omega = p(\beta)(\omega), e_\omega = l_\chi(e)(\omega), a_\omega = l_\chi(a)(\omega)$. As $a^*_\omega a_\omega$ is a positive element of the C^*-algebra $U(\omega)$,the number $\|a^*_\omega a_\omega\|_{U(\omega)}$ belongs to the spectrum $Sp(a^*_\omega a_\omega)$ of the element $a^*_\omega a_\omega$.So we have the inequality

$$|\alpha_\omega + \beta_\omega\|a_\omega\|^2_{U(\omega)}| \leq sup\{|\alpha_\omega + \beta_\omega\lambda_\omega| : \lambda_\omega \in Sp(a^*_\omega a_\omega)\}.$$

By the formula for the spectral radius of the normal element $\alpha_\omega e_\omega + \beta_\omega a^*_\omega a_\omega \in U(\omega)$ we get

$$sup\{|\alpha_\omega + \beta_\omega\lambda_\omega| : \lambda_\omega \in Sp(a^*_\omega a_\omega)\} = \|\alpha_\omega e_\omega + \beta_\omega a^*_\omega a_\omega\|_{U(\omega)}$$

.

Consequently $|\alpha_\omega + \beta_\omega \|a_\omega\|^2_{U(\omega)}| \leq \|\alpha_\omega e_\omega + \beta_\omega a^*_\omega a_\omega\|$. Therefore and from the equality $a^*_\omega = l_\chi(a^*)(\omega)$ we get that $|\alpha + \beta\|a\|^2| \leq \|\alpha e + \beta a^* a\|$. Which means that f is $L^\infty(\Omega)$-bounded on B and $\|f\| \leq 1$. But $f(e) = \mathbf{1}$, consequently, $\|f\| = \mathbf{1} = f(e)$. Due to the Hahn-Banach-Kantorovich theorem f has an extension g onto U , in addition, $\|g\| = \mathbf{1} = f(e) = g(e)$. This means that g is a state on U (see **Proposition 3.1**) and $g(a^*a) = f(a^*a) = \|a\|^2$.

Let $K_a(U)$ be a set of states ψ such that $\psi(a^*a) = \|a\|^2$. Clearly, $K_a(U)$ is a nonempty convex cyclic *-weakly closed subset of E_U . So $K_a(U)$ is an *-weakly cyclic compact, and therefore due to the vector Krein-Milman theorem ((Kusraev, 1985), P. 58) the set $K_a(U)$ has limiting points. Chose any limiting point $\varphi \in K_a(U)$ and assume that $2\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in E_U$. We have $\varphi_i(a^*a) \leq \|a\|^2, i = 1, 2$, and $2\|a\|^2 = \varphi_1(a^*a) + \varphi_2(a^*a)$. The latter is possible only if $\|a\|^2 = \varphi_1(a^*a) = \varphi_2(a^*a)$, i.e., $\varphi_1, \varphi_2 \in K_a(U)$. As φ is a limiting point of $K_a(U)$, we have $\varphi = \varphi_1 = \varphi_2$. That means φ is a limiting point of E_U, and due to **Proposition 3.2** φ is a pure state on U. □

4. Representation of Commutative C*-Algebras Over Arens Algebras

As in the previous section, we denote by $K(U)$ the set of all pure states on a unital C^*-algebra U over L^ω.

Definition 4.1. *We say that a mapping $f : K(U) \to L^\omega$ is mixing-preserving, if for an arbitrary unity partition $(\pi_n)_{n\in N}$ in ∇ and $(\varphi_n)_{n\in N} \subset K(U)$,*

$$f\left(\sum_{n=1}^{\infty} \pi_n \varphi_n\right) = \sum_{n=1}^{\infty} \pi_n f(\varphi_n)$$

For $\alpha, \beta \in L^\omega, \varphi\psi \in K(U)$ we put

$$d_{K(U)}(\varphi, \psi) = \bigwedge\{\pi^\perp : \pi\varphi = \pi\psi, \pi \in \nabla\}$$

and

$$d(\alpha, \beta) = \bigwedge\{\pi^\perp : \pi\alpha = \pi\beta, \pi \in \nabla\}.$$

Proposition 4.1. *A mapping $f : K(U) \to L^\omega$ preserves mixing if and only if $d(f(\varphi), f(\psi)) \leq d_{K(U)}(\varphi, \psi)$ for all $\varphi, \psi \in K(U)$.*

Proof. Necessity. Let $\pi = d_{K(U)}(\varphi, \psi)^\perp$. Then $\pi\varphi = \pi\psi$. For $q \in K(U)$ we have $\pi\varphi + \pi^\perp q = \pi\psi + \pi^\perp q$, and since f preserves confusions, we conclude that $\pi f(\varphi)+\pi^{\perp}f(q)=\pi f(\psi)+\pi^{\perp}f(q)$. Hence $\pi f(\varphi)=\pi f(\psi)$, and therefore $d(f(\varphi),f(\psi))\leq\pi^{\perp}$. Consequently,

$$d(f(\varphi),f(\psi))\leq d_{K(U)}(\varphi,\psi).$$

Sufficiency. If $\varphi=\sum_{n=1}^{\infty}\pi_n\varphi_n$, then $\pi_n\varphi=\pi_n\varphi_n, n\in N$, and therefore

$$\pi_n\leq d^{\perp}_{K(U)}(\varphi,\varphi_n)\leq d^{\perp}(f(\varphi),f(\varphi_n))$$

i.e., $\pi_n f(\varphi)=\pi_n f(\varphi_n)$ for all $n\in N$. This means that

$$f\left(\sum_{n=1}^{\infty}\pi_n\varphi_n\right)=\sum_{n=1}^{\infty}\pi_n f(\varphi_n).$$

We say that mappings, satisfying the inequality in **Proposition 4.1**, do not extend the ∇-metric (Abasov and Kusraev, 1987).

Consider on $K(U)$ an *-weak topology induced from U^*. Let $C_m(K(U),L^{\omega})$ stand for the set of all continuous, mixing-preserving mappings from $K(U)$ into L^{ω}. For each $f\in C_m(K(U),L^{\omega})$ the set $f(x):x\in K$ is a cyclic compact in L^{ω}, and therefore it is order bounded in L^{ω}. Consequently, an element $\|f\|=sup|f(x)|:x\in K(U)$ of L^{ω} is defined. Consider in $C_m(K(U),L^{\omega})$ pointwise algebraic operations and the involution. □

Proposition 4.2. $(C_m(K(U),L^{\omega}),\|\cdot\|)$ *is a* C^**-algebra over* L^{ω}.

Proof. In view of **Proposition 4.1**, $C_m(K(U),L^{\omega})$ coincides with the set of all continuous mappings from $K(U)$ into L^{ω} which do not extend the ∇-metric. According to Abasov and Kusraev (1987) (theorem 2), $(C_m(K(U),L^{\omega}),\|\cdot\|)$ is a Banach-Kantorovich space over L^{ω}. The definition of the norm immediately implies that $\|f\cdot g\|\leq\|f\|\|g\|$ and $\|\overline{f}\cdot f\|=\|f\|^2$ for all $f,g\in C_m(K(U),L^{\omega})$. □

The following result is a vector statement of the classical Gelfand–Naimark theorem.

Theorem 4.1. *Let* U *be a unital commutative* C^**-algebra over* L^{ω} *and let* $K(U)$ *be the set of all pure states on* U*. Then* U *is isometrically *-isomorphic to* $C_m(K(U),L^{\omega})$.

Proof is similar to that of the classical Gelfand–Naimark theorem.

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